Numerical Computation of Orbit Error Covariance Functions

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Introduction

TRUNCATION of the Earth's gravitational field, when using a well-known low-degree model, and the uncertainty in the atmospheric drag parameters are two main force model errors that affect the ephemerides of low-altitude artificial satellites generated by numerical orbit propagation procedures. A statistical estimation of these orbit errors is very useful in defining an uncertainty ellipse of the region where a satellite would rise within the visibility circle of a tracking station. There exist some stochastic procedures to evaluate the order of magnitude of the accumulated global errors in short-term[1] as well as in long-term[2,3] orbit error propagations. The resulting orbit error covariance functions are bidimensional in nature, and a fast and accurate numerical quadrature is usually required for their evaluation. This work aims at determining a suitable quadrature for the evaluation of these double integrals. After explaining about various quadratures and their performance in some simple problems, results obtained from actual satellite data are presented. Orbit errors caused by numerical integrators have been dealt with elsewhere.[4]

Orbit Error Covariance Function

Using the set $\{r, v, \beta, I, \Omega, t\}$ of spherical elements,[1] where r is the radial distance between the satellite and the Earth's center, v the satellite velocity, β the angle between the satellite radius vector and the velocity vector, I the orbital inclination, Ω the longitude of ascending node, and t the time, for defining a 6×1 orbit state vector $\boldsymbol{X}(\zeta)$, where ζ is the angle between the maximum declination point and the instantaneous position of the satellite, the general expression of the covariance function of $\Delta\boldsymbol{X}(\zeta)$ can be expressed as

$$\begin{aligned}\operatorname{cov}\{\Delta\boldsymbol{X}(\zeta)\} &= \phi(\zeta,\zeta_0)E\left\{\Delta\boldsymbol{X}(\zeta_0)\Delta\boldsymbol{X}^T(\zeta_0)\right\}\phi^T(\zeta,\zeta_0)\\ &+\int_{\zeta_0}^{\zeta}\int_{\zeta_0}^{\zeta}\phi(\zeta,\tau)\boldsymbol{G}(\tau)E\{\Delta\boldsymbol{u}(\tau)\Delta\boldsymbol{u}^T(\eta)\}\\ &\times\boldsymbol{G}^T(\eta)\phi^T(\zeta,\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \qquad (1)\end{aligned}$$

where ϕ is a 6×6 orbit error transition matrix, G a 6×3 control matrix and $E\{\Delta\boldsymbol{u}(\tau)\Delta\boldsymbol{u}^T(\eta)\}$ the force model error covariance function with $\Delta\boldsymbol{u}$ as a zero-mean Gaussian process. The double integral, which considers the effect of the force model errors, is an additive deweighting matrix.

Quadratures used in this study are YD1024,[5] a trapezoidal rule with Romberg principle of extrapolation, with choice of upper limit for absolute error and of number of subintervals and with an indicator if the accuracy required is attained; DBLINT,[6] an adaptive Romberg method, with desired absolute error as input and estimated bound as output parameters and with indicator if integration is complete; quadrature, adaptive Newton Cotes' 8 panel (QUANC8),[7] that integrates an eighth-degree interpolating polynomial over equally spaced evaluation points getting exact results for polynomials of degree 9; D01DAF,[8] Patterson's optimum addition of points to Gauss quadrature formulas, using a family of interlacing common point formulas (from 3 points up to 255 points) and with choice of absolute accuracy; SIMPS,[9] a composite Simpson's rule (repeated application of the low-order Simpson's formula); TWODQ,[10] a Gauss–Kronrod rule with either one of six pairs (from 7–15 point pair to 30–61 point pair), with desired absolute and relative accuracies as input parameters and estimated absolute error as output parameter; and QGAUSS and GAUSS6,[11] Gauss–Legendre formulas, the former an N-point quadrature with variable N and the latter with repeated application of a 6-point rule, in each interval.

Test Problems

The main test case chosen is the circular orbit of a satellite at 400-km altitude. Circular orbits at 600- and 800-km altitudes are also considered in some cases. Using a geopotential function model complete to degree and order 30 as a validation model and a geopotential model complete to degree and order 6 as a working model, the true error in the propagation has been generated by using a numerical generator.[12] Then, considering the gravity error covariance function $E\{\Delta u(\tau)\Delta u^T(\eta)\}$ consistent with the force model error assumed, the double integral given in Eq. (1) has been evaluated in various subintervals of a single orbit. Two computers, Burroughs-6800 and VAX-11/780, with machine epsilons (magnitude of the smallest finite quantity that can be represented in the machine) of 1.0E-11 and 1.0E-14, respectively, are used here since some quadratures are available in only one of these computers.

Results and Discussion

The performance analysis of all of the quadratures in the main test case is shown in Table 1. Results with YD1024,[5] used with an error tolerance limit of 1.0E-03 in a 0.1 orbit with 20 subintervals of integration, are not bad. But, the CPU time for this 9-min orbit is 7 min. In consequence, it is discarded. DBLINT[6] provided good error estimates in 0.1 orbit but with high CPU time (15 min) and so this quadrature also is discarded. QUANC8[7] provided promising results in 0.1 and 0.2 orbit intervals. In 0.5 orbit (45 min), the estimates are good but the CPU time is high (44 min). Hence, it has been rejected. In 0.1 orbit, as well as in 0.5 orbit, D01DAF[8] gave good error estimates in relatively low CPU times. In other integration intervals also the results are consistent. But, in a 800-km altitude circular orbit, the integration is interrupted due to some fatal error. This made the routine unreliable. Results obtained with SIMPS,[9] making number of applications of Simpson's rule in a given interval as a function of the length of the interval, are good, and conservative and CPU times also are tolerable. In fact, the results presented previously[1] were obtained by using this quadrature only. Performance of TWODQ[10] is, in general terms, similar to that of D01DAF. Besides, it did not cause any problem in other example problems. Hence, the usage of this quadrature is left open for discussion. For QGAUSS, abscissas and weights are taken in the interval $[-1, +1]$. Dividing the orbit into 10 intervals, when considering ni intervals for integration each time, a 6*ni-point Gauss rule is applied in that interval. It is found that the formula for the number of function evaluations in each call of QGAUSS with 6*ni-points is (6*ni)**2. In each interval, for the total of six elements, the number of total function evaluations is 6*[(6*ni)**2]. There is no automatic error control embedded in the subroutine. The results obtained here, in terms of accuracy and also in terms of CPU time, are simply excellent. CPU time for 0.1 orbit is less than 6 s, for 0.5 orbit it is 2.3 min, and for a full orbit (90-min period) it is 9.5 min. Consequently, this quadrature is considered to be the best of all quadratures considered until now. Finally, for GAUSS6,[11] the

Table 1 Performance analysis of the quadratures in a 400-km orbit

Quadrature and computer	0.1 Orbit Accuracy	0.1 Orbit CPU	0.5 Orbit Accuracy	0.5 Orbit CPU, min
YD1024, Burroughs	Not bad	7 min	——	——
DBLINT, Burroughs	Good	15 min	——	——
QUANC8, Burroughs	Good	9 min	Good	44
D01DAF, Burroughs	Good	40 s	Good	20
SIMPS, VAX-11/780	Good	1.25 s	Good	22
TWODQ, UTXVMS	Good	50 s	Good	17
QGAUSS, VAX-11/780	Good	6 s	Good	2.3
GAUSS6, VAX-11/780	Good	6 s	Good	2.3

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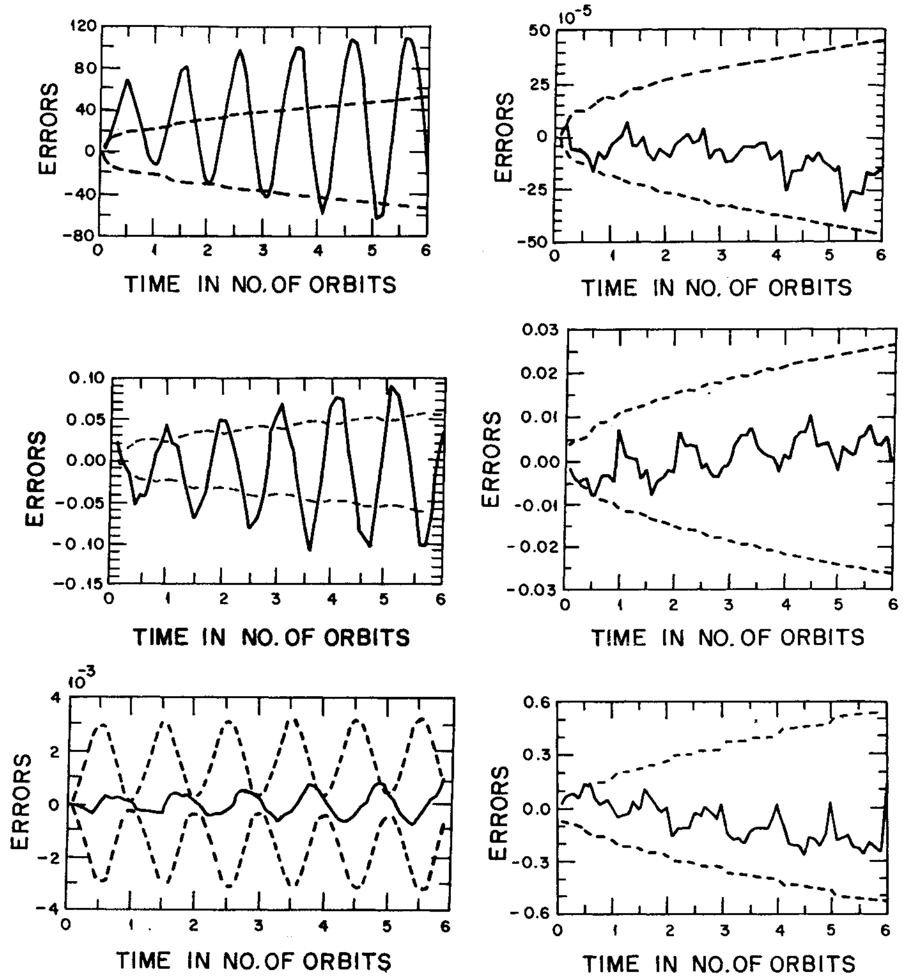

Fig. 1 Comparison of the orbit errors for the first Brazilian satellite SCD1.

abscissas and weights are taken in the interval [0,1]. Dividing the orbit into 10 intervals, when considering *ni* intervals for integration, the Gauss 6-rule is applied *ni* times repeatedly. The formula for the number of function evaluations in each call of GAUSS6 is found to be (6**2)(*ni***2). In each interval, for the total of six elements, the number of function evaluations is 6*(6**2)*(*ni***2). There is no automatic error control embedded in the subroutine. In terms of the number of function evaluations involved, the accuracy obtained, and the CPU time, the performance of GAUSS6 is equivalent to that of QGAUSS. In 600- and 800-km circular orbits also the results are equally good. Encouraged by the results obtained, QGAUSS is used in the case of the first Brazilian satellite SCD1 that was launched on Feb. 9, 1993. Its orbit is near circular with an average altitude of 760 km. In the general expression for drag, the uncertainties in the atmospheric density, in the ballistic coefficient, and in the atmospheric rotation parameter are considered; the covariance matrices of these errors are defined, and another additive deweighting matrix that considers these errors is added in Eq. (1). Figure 1 shows the results obtained in a long-term propagation (half-dozen orbits) for the combined effect. Solid lines in the figure are the true errors and the dashed lines are σ variation of the estimated errors.

Conclusion

Gauss–Legendre quadrature, either with repeated application of a fixed N-point rule (GAUSS6) or with number of points varying with length of the interval (QGAUSS), is the best quadrature available at present for computing bidimensional orbit error covariance integrals.

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